General Physics: Electromagnetism, Correction 9

Exercise 1 :

A long straight wire of radius a carries a current that is uniformly distributed over its cross-section. Find the magnetic field both inside and outside the wire.

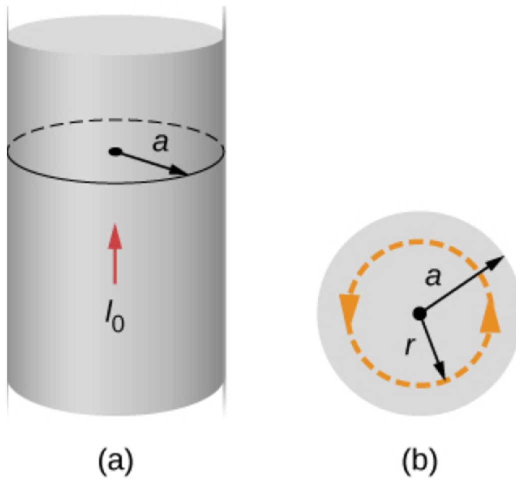


Figure 1: (a) A model of a current-carrying wire of radius a and current I_0 (b) A cross-section of the same wire showing the radius a and the Ampere's loop of radius r.

Solution 1 :

To find the magnetic field we apply the Ampere's law in its full generality

$$\oint_\Gamma d\vec{\ell} \cdot \vec{B} = \mu_0 I_{\rm enc} = \int_{\mathcal{A}} d\vec{\sigma} \cdot \vec{J}. \tag{1}$$

We have to distinguish two cases. For $r < R$, we have

$$I_{\rm enc} = \int_{\mathcal{A}} d\vec{\sigma} \cdot \vec{J} = \int_0^r dr' 2\pi r' \left(\frac{I_0}{\pi a^2}\right) = \frac{I_0}{a^2} r^2. \tag{2}$$

The Ampere's law gives $2\pi r B_1 = \mu_0 I_0 r^2/a^2$ and then $B_1 = \mu_0 I_0 r/2\pi a^2$.

For $r > R$, we have instead

$$I_{\rm enc} = \int_{\mathcal{A}} d\vec{\sigma} \cdot \vec{J} = \int_0^a dr' 2\pi r' \left(\frac{I_0}{\pi a^2}\right) = I_0. \tag{3}$$

The Ampere's law gives $2\pi r B_2 = \mu_0 I_0$ and then $B_2 = \mu_0 I_0/2\pi r$.

Exercise 2 :

Consider an infinitely long, cylindrical conductor of radius R carrying a current I with a non-uniform current density

$$J = \alpha r \tag{4}$$

where α is a constant. Find the magnetic field everywhere.

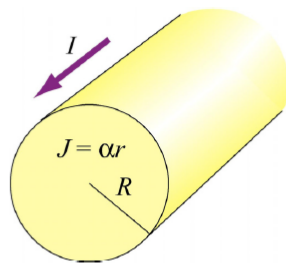


Figure 2: Non-uniform current density

Solution 2 :

To find the magnetic field we apply again the Ampere's law in its full generality

$$\oint_\Gamma d\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{enc}} = \int_\mathcal{A} d\vec{\sigma} \cdot \vec{J}. \tag{5}$$

We have to distinguish two cases. For $r < R$, we have

$$I_{\text{enc}} = \int_\mathcal{A} d\vec{\sigma} \cdot \vec{J} = \int_0^r dr' 2\pi r'(\alpha r') = \frac{2}{3}\pi\alpha r^3. \tag{6}$$

The Ampere's law gives $2\pi r B_1 = \frac{2}{3}\mu_0\pi\alpha r^3$ and then $B_1 = \alpha\mu_0 r^2/3$.

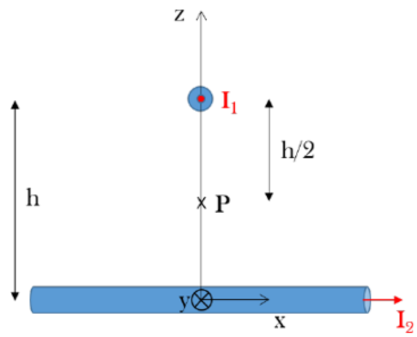
For $r > R$, we have instead

$$I_{\text{enc}} = \int_\mathcal{A} d\vec{\sigma} \cdot \vec{J} = \int_0^R dr' 2\pi r'(\alpha r') = \frac{2}{3}\pi\alpha R^3. \tag{7}$$

The Ampere's law gives $2\pi r B_2 = \frac{2}{3}\mu_0\pi\alpha R^3$ and then $B_2 = \alpha\mu_0 R^3/3r \propto 1/r$, as in the previous exercise.

Exercise 3 :

Two long wires of radius a are perpendicularly oriented as shown in figure below. The upper wire has a current I_1 in the $\hat{y}$ direction and the lower cable has a current I_2 in the $\hat{x}$ direction.



(a) Find the magnetic field along the z axis, between $z = a$ and $z = h$.

(b) For $I_1 = 100$ A and $I_2 = 150$ A, with the distance $h = 2.5$ cm, what is the magnitude of the magnetic field at point P?

(c) Describe the direction of the compass needle placed in point P.
Hint: The magnetic field of Earth is about $5 \cdot 10^{-5}$ T.

Solution 3 :

(a) From Ampere's law:

$$\oint_\Gamma d\vec{\ell} \cdot \vec{B} = \mu_0 I \tag{8}$$

we see that each wire will generate a magnetic field. For $a < z < h - a$ (i.e the space between the two wires), along the z axis, the magnetic field of each wire is:

$$B_1 2\pi(h - z) = \mu_0 I_1 \tag{9}$$

$$B_2 2\pi z = \mu_0 I_2 \tag{10}$$

With the right hand rule, we can determine the direction of each magnetic field along the z axis:

$$\vec{B}_1 = B_1 \hat{e}_x \tag{11}$$

$$\vec{B}_2 = -B_2 \hat{e}_y \tag{12}$$

Hence, the total magnetic field along the z axis is:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{2\pi} \left(\frac{I_1}{h - z} \hat{e}_x - \frac{I_2}{z} \hat{e}_y \right) \tag{13}$$

For $h - a < z < h$ the expression of $\vec{B}_2$ does not change. However, the expression for $\vec{B}_1$ changes. $\vec{B}_1$ is generated by only a part of the current I_1.

$$B_1 2\pi(h-z) = \frac{\mu_0}{a^2} I_1 (h-z)^2. \tag{14}$$

Therefore we find that the total magnetic field along the z axis inside the upper wire is:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{2\pi}\left(\frac{I_1(h-z)}{a^2}\hat{e}_x - \frac{I_2}{z}\hat{e}_y\right) \tag{15}$$

(b) Point P is found at $z = h/2$, between the two wires. Therefore, the magnetic field is found by inserting $z = h/2$ in equation (15):

$$\vec{B} = \frac{\mu_0}{\pi h}(I_1\hat{e}_x - I_2\hat{e}_y) \tag{16}$$

Finally, the magnitude of the magnetic field is:

$$\left|\vec{B}\right| = \frac{\mu_0}{\pi h}\sqrt{I_1^2 + I_2^2} = 2.88 \cdot 10^{-3}\ T \tag{17}$$

(c) Earth's magnetic field ($5{\cdot}10^{-5}\ T$) is around 100 times weaker than the magnetic field generated by the wires at point P ($2.88 \cdot 10^{-3}\ T$). The compass needle will therefore follow the magnetic field generated by the wires rather than Earth's magnetic field. At point P, the needle will be oriented at an angle $\theta = \arctan(B_y/B_x) = -56°$ with respect to the x axis in the x-y plane.

Exercise 4 :

Consider two infinitely long wires carrying currents are in the x direction.

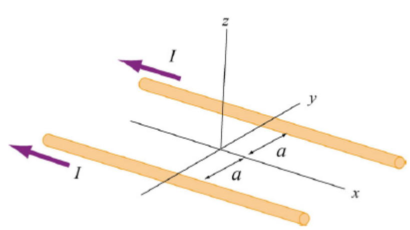


Figure 3: Non-uniform current density

(a) Draw a schematic of the magnetic field pattern in the yz-plane.

(b) Find the distance d along the z-axis where the magnetic field is maximum.

Solution 4 :

(a) The magnetic field lines are shown in Figure 4. Notice that the directions of both currents are into the page.

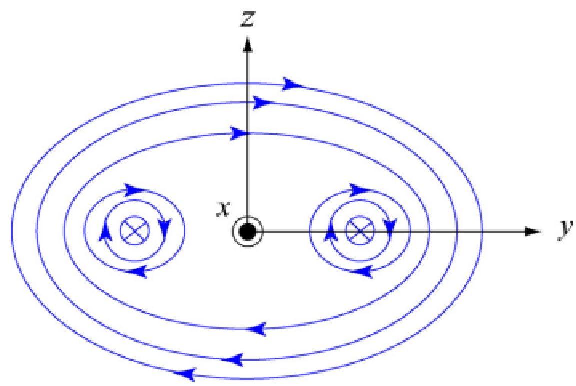


Figure 4: Magnetic field lines of two wires carrying current in the same direction.

(b) We have to compute the total magnetic field at the point $(0, 0, z)$. Let's start from wire 1, which is placed on the left. We can use the Ampere's law to find the magnitude of the magntic field

$$B_1 = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}}. \tag{18}$$

To find the direction, we have to compute the following cross product

$$(-\hat{e}_x) \times \vec{r}_1 = (-\hat{e}_x) \times (\hat{e}_y \cos\theta + \hat{e}_z \sin\theta) = \hat{e}_y \sin\theta - \hat{e}_z \cos\theta, \tag{19}$$

being θ the angle between the wire and the point $(0, 0, z)$, and $\vec{r}_1$ the vector connecting the two points. Thus,

$$\vec{B}_1 = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}}(\hat{e}_y \sin\theta - \hat{e}_z \cos\theta). \tag{20}$$

With the same procedure, we can find the magnetic field on $(0, 0, z)$ generated by the second wire

$$\vec{B}_2 = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}}(\hat{e}_y \sin\theta + \hat{e}_z \cos\theta). \tag{21}$$

The total magnetic field reads

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I \sin\theta}{2\pi\sqrt{a^2+z^2}}\hat{e}_y = \frac{\mu_0 I z}{2\pi(a^2+z^2)}\hat{e}_y = B(z)\hat{e}_y, \tag{22}$$

since $\sin\theta = z/\sqrt{a^2+z^2}$.

To compute the distance d along the z-axis where the magnetic field is maximum we take the derivative of B with respect to z

$$0 = \frac{\partial}{\partial z}B(z) = \frac{\mu_0 I}{\pi}\frac{a^2-z^2}{(a^2+z^2)^2}, \tag{23}$$

so that $z = a$ and the magnetic field is $B_{\max} = \mu_0 I/2\pi a$.

Exercise 5 :

Consider a toroid as shown in Figure 5 with a big radius R and a small radius a. The toroid contains N turns of the wire with current I. Suppose that the number of turns N is huge, such that we can consider cylindrical symmetry. Calculate the magnetic field $\overrightarrow{B}$ for:

(a) $r < R - a$;

(b) $R - a < r < R + a$;

(c) $r > R + a$;

and draw the lines of the field. What is the magnitude of the magnetic field in $r = R$ if $I = 500$ A, $N = 100$ and $R = 0.5$ m?

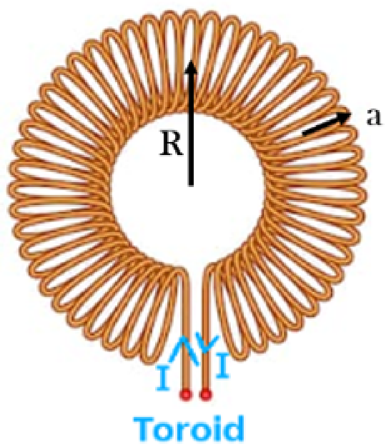


Figure 5: Toroid

Solution 5 :

A toroid can be seen as a solenoid bent in the shape of a toroid. With this assumption as well as the cylindrical symmetry one, one could already deduce that the field is only inside the toroid as seen in theory.

Let's now concretely take the a), b) and c) cases individually and calculate the magnetic field. To better visualize this, a horizontal section of the toroid (simplified) will be considered (Figure 6):

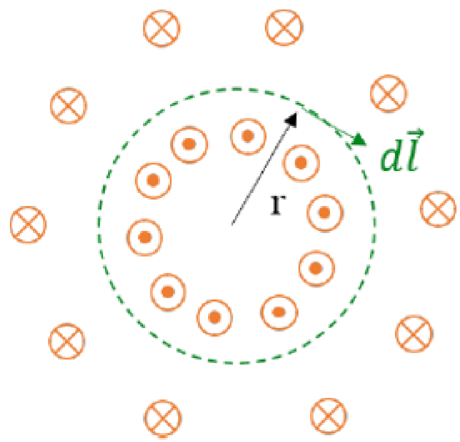


Figure 6: Toroid section

From Ampere's law, stating that magnetic field inside a given area is a result of the enclosed current in this same area, one can conclude that when $r < R - a$ (case a)), their is no enclosed current and thus the magnetic field B is equal to zero as well. At the opposite case, when $r > R+a$ (case c)), outside of the toroid, as for each current entering there is an equal in norm but opposite in direction outgoing current (this can easily be understood from Figure 6), the total current enclosed is also zero and thus magnetic field is also zero. Now, when $R - a < r < R + a$ (case c), which is the case depicted in Figure 6, we have an enclosed current $I_{\text{enclosed}} = NI$ non zero because only the inner part of the toroid and its currents are inside the loop. As these currents are all having the same direction (and intensity), they just add up. This time the Ampere's law has to be concretely applied:

$$\oint_{\Gamma} d\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{enclosed}} \tag{24}$$

That we can rewrite due to our two initial assumptions and our formulation for I_{enclosed}:

$$B(r) 2\pi r = \mu_0 N I \tag{25}$$

Thus between $R - a$ and $R + a$, we have a magnetic field $B(r)$ equal to:

$$B(r) = \frac{\mu_0 N I}{2\pi r} \tag{26}$$

The numerical result with the given I, r and N, also knowing the vacuum magnetic permeability $\mu_0 = 4\pi \times 10^{-6}$ N/A^2, is $B \simeq 0.02$ T.

To end up with, the field lines drawing should look like this:

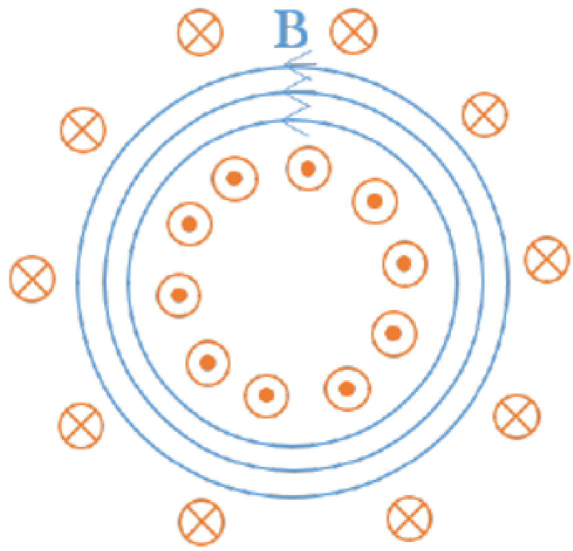


Figure 7: Field lines

Exercise 6 :

Find the minimum diameter of the wires d that can transmit $P = 225$ MW of electricity with only a 2.0% loss. Their length is $l = 185$ km. Assume there are two wires to make a complete circuit (the length is thus doubled). The wires are to be made of aluminum ($\rho = 2.6 \cdot 10^{-8}\ \Omega \cdot m$) and the voltage is $V = 660$ kV.

Solution 6 :

The minimum diameter is the one which gives, from the 2nd Ohm's law, a value of resistance R which introduces 2.0% of losses:

$$R = \rho \frac{l_{\text{tot}}}{A} = \rho \frac{2l}{\pi (d/2)^2}, \tag{27}$$

since, as specified in the text, the two wires form a loop of length $2l$.
The dissipated power is $P_{\text{dis}} = RI^2 = \alpha P$, where $P = VI$ is the transmitted power and $\alpha = 0.02$ is the loss factor. So, we can write:

$$P_{dis} = \alpha P = RI^2 = R\frac{P^2}{V^2}, \tag{28}$$

which gives:

$$\alpha P = R\frac{P^2}{V^2} \rightarrow R = \frac{\alpha V^2}{P}. \tag{29}$$

By solving Eq. (27) for d and inserting R from Eq. (29), we get

$$d = \sqrt{\frac{8\rho l}{\pi R}} = \sqrt{\frac{8\rho l P}{\pi \alpha V^2}} = \frac{1}{V}\sqrt{\frac{8\rho l P}{\pi \alpha}} = 18 \text{ cm}. \tag{30}$$